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A GENERALIZATION OF CAUCHY'S FUNCTIONAL EQUATIONS.*

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§ 1. INTRODUCTION.

Cauchy† has determined the general continuous solution of each of the simple functional equations

$$\begin{aligned} f(x+y)&=f(x)+f(y),\\ f(x+y)&=f(x)\,.\,f(y),\\ f(xy)&=f(x)\,.\,f(y),\\ f(xy)&=f(x)+f(y), \end{aligned}$$

showing that they are, respectively, ax, a^x, x^a, $a\log x$, where in each case a is an arbitrary constant.

Several generalizations of these results have been given. For instance, in the case of the first equation it has been shown‡ that the conclusion remains the same if the hypothesis is weakened by requiring that $f(x)$ shall be continuous only in a finite interval, however small. Or, one may require§ merely that the function shall have an upper limit in an arbitrarily small neighborhood of $x=0$. A generalization of a different kind has been given by H. W. Pexider.||

In the present paper a generalization of a different type is introduced. Instead of two variables, x and y, of unlimited variation, we retain one variable x of unlimited variation and replace the variable y by a function u_z of the variable z which has the property that, for some set of values of z, u_z may assume at least once each value of an assigned enumerably infinite set of values. Otherwise the function u_z is entirely arbitrary; it need not even be defined at other points. Furthermore, $f(x)$ is required to be continuous only in a given finite interval. The general solution remains the same as before.

This generalization is interesting in that it brings out explicitly the fact that in previous discussions of the problem, as for instance by Cauchy and Darboux, the hypothesis was much stronger than was necessary to enable one to draw the conclusion.

The method employed in the present note (see § 2) is capable of wide use and extension in the study of functional equations of various types. It is due to Monge and Laplace. It consists, abstractly, in reducing the original problem to two steps. The first step consists in finding the general so-

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* Read before the American Mathematical Society, October 28, 1911.

† Cauchy, *Œuvres* (2) 3 (1897), p. 98, p. 220.

‡ Darboux, *Mathematische Annalen* 17 (1880), p. 55; Segre, Torino Atti 25 (1890), p. 192. p. 287.

§ La Vallée-Poussin, *Course d'analyse*, 1903, p. 30.

|| *Monatshefte fur Mathematik und Physik*, 14 (1902), p. 293.

lution of a difference equation which is a special case of the given equation, or is readily deducible from it, this equation having the property that some particular solution of it is the general solution of the original functional equation. The second step consists in determining the arbitrary periodic function or functions of this solution of the difference equation so as to obtain that special solution which satisfies the given functional equation. A like use may also be made of q-difference equations in the solution of functional equations.

In the present paper all quantities are supposed to be real. The extension to the case of complex quantities is not difficult.

§ 2. THE EQUATION $f(x+u_z)=f(x)+f(u_z)$.

We seek to determine completely the function $f(x)$ subject to the conditions that it is to be continuous in the interval 0 to α, where α is any positive or negative constant whatever, and that it is to satisfy the functional equation

$$(1) \qquad f(x+u_z)=f(x)+f(u_z),$$

where u_z is a function of the variable z subject to the sole restriction that it is capable of assuming at least once each value in the set

$$2^{-s}n\,\alpha; \quad n,\ s=0,\ 1,\ 2,\ \ldots,\quad n \leqq 2^s,$$

$u_z=z$ is the simplest such function; and hence equation (1) is a generalization of the Cauchy equation

$$f(x+z)=f(x)+f(z).$$

But the possible functions u_z are evidently of the most varied character. The most general definition of u_z is the following:

Let z_{ns}, n, $s=0, 1, 2, \ldots, n \leqq 2^s$, be any set of numbers no two of which are equal and let

$$u_z=\frac{n\,\alpha}{2^s} \text{ for } z=z_{ns};$$

otherwise let u_z be arbitrary. It need not even be defined for other values of z. Under these conditions we shall show that the solution of (1) is unique except for an arbitrary constant multiplier; in fact, that it is

$$f(x)=ax,$$

where a is an arbitrary constant.

Give to z a value for which u_z is equal to α; then from (1) we have the difference equation

$$f(x+\alpha)=f(x)+f(\alpha). \tag{2}$$

Therefore the general solution of (1) must be some particular solution of (2). But the general solution of (2) is readily found. It is evidently

$$f(x)=ax+p(x) \tag{3}$$

where $a=f(\alpha)/\alpha$ and $p(x)$ is an arbitrary periodic function of period α. Since $f(x)$ is continuous in the interval $(0, \alpha)$, $p(x)$ is continuous in $(0, \alpha)$. Substiiuting this value of $f(x)$ in (1) and reducing, we have

$$p(x+u_z)=p(x)+p(u_z), \tag{4}$$

an equation of the same type as (1) but subject to the further condition that its solution is to be periodic of period α.

Choose z so that $u_z=0$ and take $x=0$. Then from (4) we have

$$p(0)=2p(0);\ \text{or}$$
$$p(0)=0,$$

since $p(x)$ is finite at $x=0$. Hence

$$0=p(0)=p(\alpha)=p(2\alpha)=\ldots=p(n\alpha).$$

Give to z a value such that u_z takes on the value $2^{-s}n\alpha$ and let $x=2^{-s}n\alpha$. Then from (4) we have

$$p\left(\frac{n\alpha}{2^{s-1}}\right)=2p\left(\frac{n\alpha}{2^s}\right).$$

If we employ this as a recursion formula and remember that $p(n\alpha)=0$, we have

$$0=p\left(\frac{n\alpha}{2}\right)=p\left(\frac{n\alpha}{2^s}\right)=\ldots=\left(\frac{n\alpha}{2^s}\right).$$

That is, the function $p(x)$ is equal to zero at every point of the interval

$(0, \alpha)$ which is representable in the form $2^{-s}n\,\alpha$. But the set of points $2^{-s}n\,\alpha$, $n, s=0, 1, 2, \ldots, n \leqq 2^s$ is everywhere dense in the interval $(0, \alpha)$; that is, in every neighborhood (however small) of any point $\bar{\alpha}$ of the interval $(0, \alpha)$ there is a point of the set $2^{-s}n\,\alpha$ different from $\bar{\alpha}$.

Now, if a function is continuous at every point of an interval $(0, \alpha)$ and is equal to zero at every point of a set of points everywhere dense on $(0, \alpha)$, it is clear that the function is equal to zero at every point of the interval $(0, \alpha)$. Hence $p(x)=0$ at every point of the interval $(0, \alpha)$. But $p(x)$ is a periodic function of period α; and therefore $p(x)$, being zero in an interval of length α, is identically zero. Substituting this value of $p(x)$ in (3) we have as the most general solution of (1),

$$f(x)=ax,$$

where a is an arbitrary constant. This result may be stated in the form of the following theorem:

THEOREM I. *Let* α *be any positive or negative qaantity and let* u_z *be any function of* z *capable of assuming at least once each value in the set*

$$2^{-s}n\,\alpha,\ n,\ s=0,\ 1,\ 2,\ \ldots,\ n \leqq 2^s.$$

Then the most general function $f(x)$ *which is continuous on the interval* $(0, \alpha)$ *and satisfies the functional equation*

$$f(x+u_z)=f(x)+f(u_z)$$

is

$$f(x)=ax,$$

where a *is an arbitrary constant.*

§ 3. THE REMAINING CAUCHY FUNCTIONAL EQUATIONS.

The results of the last section we shall now apply to the problem of finding the most general solution of each of the following equations:

$$f(x+u_z)=f(x)f(u_z), \tag{5}$$

$$f(xu_z)=f(x)f(u_z), \tag{6}$$

$$f(xu_z)=f(x)+f(u_z), \tag{7}$$

where the functions are further restricted in each case to satisfy appropriate conditions of continuity on a limited interval. As a special value of u_z we may take $u_z=z$, so that (5), (6), and (7) are generalizations of Cauchy's equations

$$f(x+z)=f(x)f(z),$$
$$f(xz)=f(x)f(z),$$
$$f(xz)=f(x)+f(z).$$

If we take the logarithm of each member of (5) and write $g(v)$ for $\log f(v)$, we have

$$g(x+u_z)=g(x)+g(u_z).$$

Now if $g(x)$ is to be continuous in the interval $(0, \alpha)$ the conditions of theorem I are satisfied for the last equation, so that the general solution is

$$g(x)=ax,$$

where a is an arbitrary constant. Hence

$$\log f(x)=ax.$$

Therefore, $$f(x)=e^{ax}=\beta^x,$$

where β is an arbitrary constant. If $g(x)$ is continuous in the interval $(0, \alpha)$, $f(x)$ is continuous in the same interval, and conversely, since $f(x)$ cannot be zero on the interval without being identically zero (as the original equation shows). Hence from theorem I we have the following theorem:

THEOREM II. *If* α *and* u_z *are defined as in theorem I, then the most general function* $f(x)$ *which is continuous on the interval* $(0, \alpha)$ *and satisfies the functional equation*

$$f(x+u_z)=f(x)f(u_z)$$

is $$f(x)=\beta^x,$$

where β *is an arbitrary constant.*

In equation (6) let us write

$$x=e^y,\quad u_z=e^{v_z},\quad f(e^t)=g(t);$$

then we have $$f(e^{y+v_z})=f(e^y)f(e^{v_z});$$

or $$g(y+v_z)=g(y)g(v_z).$$

If we assume that v_z satisfies the conditions imposed on u_z in the two previous cases, and that $g(y)$ is continuous in the interval $(0, \alpha)$, we have the hypotheses of theorem II reproduced. Consequently that theorem can be applied to the last equation. If this is done and if the result is stated in terms of the original function f and the variables x and u_z we have the theorem:

THEOREM III. *Let* α *be any positive or negative quantity and let* u_z *be any function of* z *capable of assuming at least once each value in the set*

$$e^{2^{-s}n\alpha}, \quad n,\ s=0,\ 1,\ 2,\ \ldots,\ n \overset{=}{<} 2^s.$$

Then the most general function $f(x)$ *which is continuous on the interval* $(1, e^{\alpha})$ *and satisfies the functional equation*

$$f(xu_z)=f(x)f(u_z)$$

is

$$f(x)=x^r,$$

where r *is an arbitrary constant.*

For, the most general value of $g(y)$ is

$$g(y)=\beta^y.$$

Hence,

$$f(e^y)=\beta_y, \text{ since } g(y)=f(e^y).$$

Now put $x=e^y$, whence $y=\log x$; then we have

$$f(x)=\beta^{\log x}=e^{r\log x},$$

where $r=\log\beta$. But $e^{r\log x}=e^{\log x^r}=x^r$. Hence the most general value of $f(x)$ is

$$f(x)=x^r,$$

where r is an arbitrary constant.

If now we take the logarithm of each member of (6) and write $g(v)$ for $\log f(v)$, we have

$$g(xu_z)=g(\alpha)+g(u_z),$$

an equation of the form (7). Hence from theorem III we have at once the following theorem:

THEOREM IV. *If* α *and* u_z *are defined as in theorem III, then the most general function of* $f(x)$ *which is continuous on the interval* $(1, e^{\alpha})$ *and satisfies the functional equation*

$$f(xu_z)=f(x)+f(u_z)$$

is

$$f(x)=r\log x,$$

where r *is an arbitrary constant.*